# Variations on a game

GREISY WINICKI LANDMAN

## Introduction

Mathematical games provide wonderful opportunities to engage students in meaningful and rich mathematical investigations. This article describes three mathematical games suitable to be played by different groups of students, from young elementary school students, learning the basics of arithmetic and geometry, to older students making their first steps in mathematical proving.

The first game is the seminal idea because the others may be obtained from changing one of its rules. The first part of the article consists of the description of the rules of the *Cross out* game, several examples of the game, its analysis and the formulation of a winning strategy. The second and the third part of the article describe two variations of the *Cross out* game, the game *Cross out in a circle* and the game *New Cross out*. The final part of the article summarises the potential embedded in playing these games.

## Cross out

*Cross out-n* is a two-player game. The board is a row of $n$ squares numbered 1 through $n$. Each player takes a turn crossing out one number or two consecutive numbers. At the end of the game, whoever crosses out the last number of the board is the winner.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

Figure 1

### Example: Cross out-8

In *Cross out-8* the board consists of eight squares numbered 1 through 8 (see Figure 1).

The following are two examples of possible games played by player A — who always plays first — and player B.

***Example 1***

| *Player* | *Cross-out move* |
|---|---|
| A | 1 |
| B | 6, 7 |
| A | 2 |
| B | 8 |
| A | 4, 5 |
| B | 3 |

B wins

***Example 2***

| *Player* | *Cross-out move* |
|---|---|
| A | 4, 5 |
| B | 7, 8 |
| A | 6 |
| B | 1 |
| A | 2, 3 |

A wins

The examples show that both players have chance to win in this mathematical game, because B won in first example and A did in the second one; but, is there any way a player can prevent his opponent from winning?

Mathematical games are a special type of mathematical problem and the solution to this type of problem is called a winning strategy. A winning strategy is a sequence of moves that lead the player who uses it to a certain win, no matter what his opponent does. It is important to point out that a winning strategy is always composed of two rules:

a) an explicit criterion to decide whether to play as a first player or as a second one; and
b) a clear description of the moves to employ.

So, when playing these types of game, students may be led to ask and to analyse the following questions:

- Does one of the players have an advantage over the other?
- Is there a sequence of moves that leads one of the players to a certain win?
- Which sequence is it?

After playing *Cross out-8* a couple of times students may discover that in this game the first player has an advantage over his opponent. If, in his opening move, player A crosses out the numbers 4 and 5, for every move player B has, that is crossing out one number or two consecutive numbers, player A has a corresponding move on the other half of the board. For example, if in his first move player B crosses out the numbers 7 and 8, player A may cross out the numbers 1 and 2. On the contrary, if in his first move player B crosses out the number 3, player A may cross out the number 6. After this, player A is always able to copy each one of his opponent's moves.

Therefore, in this case, the winning strategy is:

a) Play as the first player and cross out the numbers 4 and 5 since the board is symmetric with respect to those cells.
b) In all the subsequent moves, cross out the corresponding number (or numbers) to those crossed out by the second player.

This "symmetric strategy" in which the positions crossed out by the players form a symmetric board after each move of the first player, leads player A to win because for every possible move of the second player, there is always a possible response for the first player, who therefore will win.

## Cross out-9

In this case, the board consists of nine squares, numbered 1 through 9.

If player A implements the strategy described before for *Cross out-8*, he will be surprised. If player A crosses out the numbers 4 and 5, player B may guarantee a win by crossing out the number 6, because now the second player is the one who controls the symmetric board. Thus, the winning strategy for the game *Cross out-8* is not a winning strategy for *Cross out-9*. Maybe in this case, there is a winning strategy for the second player? Another surprise: no, there is still a winning strategy for the first player. If in his first move player A crosses out the number 5, for each one of player B's moves, player A has a corresponding move on the other half of the board. By using the "symmetric strategy", a win is guaranteed. This idea is implemented in the following examples:

***Example 1***

| Player | Move |
|---|---|
| A | 5 |
| B | 2, 3 |
| A | 7, 8 |
| B | 9 |
| A | 1 |
| B | 4 |
| A | 6 |
| A wins | |

***Example 2***

| Player | Move |
|---|---|
| A | 5 |
| B | 8, 9 |
| A | 1, 2 |
| B | 4 |
| A | 6 |
| B | 7 |
| A | 3 |
| A wins | |

## Generalisation

The main difference between the boards of the games described before is the existence or non-existence of a middle square. In the game *Cross out-9*, the square numbered 5 divides the board into two symmetric groups, but in the board used in the game *Cross out-8* there is no such square. Instead there are a couple of such squares: those numbered 4, 5.

In both cases, there is a winning strategy for the first player, but these strategies are different: in one case, player A has to cross out the square in the middle of the board and in the other case, player A has to cross out two squares.

### Summarising

In the game *Cross out-n*, there is always a winning strategy for the first player.

- If the number of squares is odd (i.e., $2n-1$), then the first player must cross out the square in the $n$th position in order to be able to win.
- If the number of squares is even (i.e., $2n$), then the first player has to cross out the squares in the $n$th and $n$th +1 positions, in order to be able to win.

After this first move, the first player will be able to copy all the opponent's moves and, by so doing, will cross out the last number.

## Cross out in a circle

Cross out in a circle is similar to the game Cross out with a slight difference: instead of having n numbers written in a row, in this case the numbers are arranged in a circle. This game is discussed by Gardner (1961, pp. 55–56).

Let us start with the numbers 1 through 9 arranged in a circle and players taking turns crossing one number or two "consecutive" numbers. In this case, 1 and 9 are consecutive numbers (see Figure 2).

The following examples may clarify the new game.

***Example 1***

| *Player* | *Move* |
|---|---|
| A | 3, 4 |
| B | 9, 1 |
| A | 5, 6 |
| B | 7, 8 |
| A | 2 |
| A wins | |

***Example 2***

| *Player* | *Move* |
|---|---|
| A | 4 |
| B | 8, 9 |
| A | 2, 3 |
| B | 5, 6 |
| A | 7 |
| B | 1 |
| B wins | |

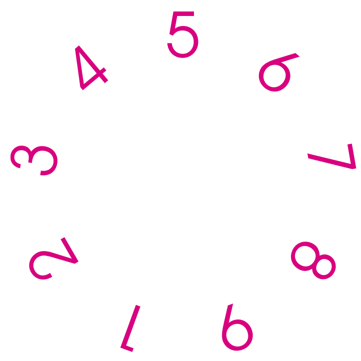


Figure 2

After playing a couple of games, you may have discovered that this game is the revenge of player B: in *Cross out in a circle*, there is a winning strategy for the second player.

For example, if player A crosses out the number 4 (see Figure 3), player B gets a board similar to the row described in Figure 4. If player B crosses out the numbers 8 and 9 he will guarantee himself a win because the game has become *Cross out* with eight numbers and the "symmetric strategy" described for that game will work.

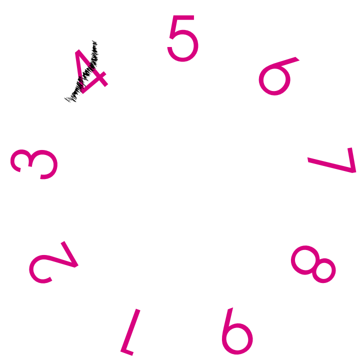


Figure 3

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 |

Figure 4

On the other hand, if player A crosses out the numbers 7 and 8, player B gets a board similar to the one shown in Figure 5. According to the winning strategy of *Cross out*,

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 1 | 2 | 3 | 4 | 5 | 6 |

Figure 4

the first player to play can guarantee a win by crossing out the number 3. So, if player B crosses out the number 3, a win is guaranteed in the game *Cross out in a circle*.

Let us play now with the numbers 1 through 8 arranged in a circle. In this case, the numbers 1 and 8 are consecutive. After the opening move of player A, the game turns into *Cross out* with six or seven numbers (see Figure 6 and Figure 7).

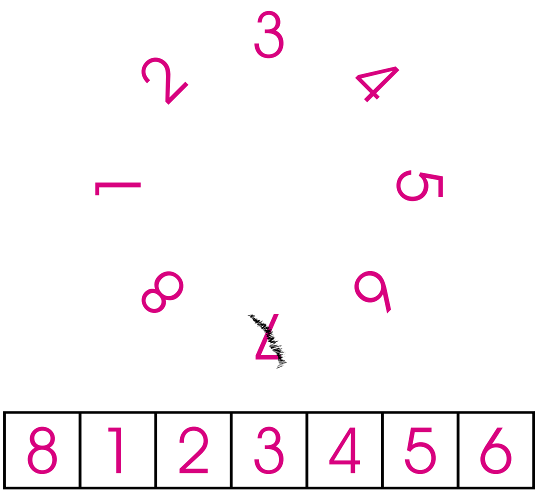
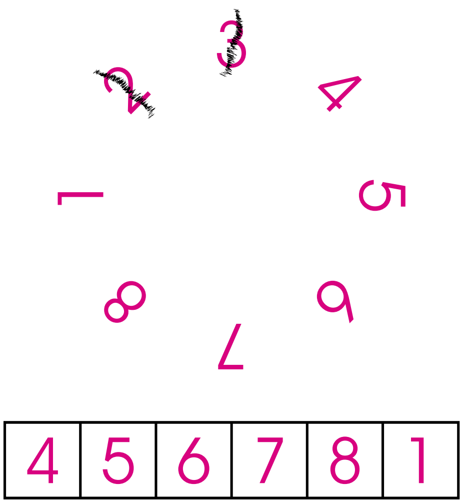



Figure 6




In the first case, player B can win by crossing out the fourth number in the row (the number 3) and in the latter case, he can win by crossing out the third and the fourth one (the numbers 6 and 7 respectively).

## Conclusion

In the game *Cross out in a circle* there is a winning strategy for the second player. The strategy is playing according to the strategy of the game *Cross out* with the board obtained after the first move of the first player.

# New Cross out

*New-Cross out* is another two-player game. The objective and the board are similar to those of the *Cross out* game. In this new version of the game, players take turn crossing out one number or two numbers that add up to an odd number. At the end of the game, whoever crosses out the last number of the board is the winner.

## New Cross out-8

The following two examples may help to clarify the rules of this new game.

***Example 1***

| *Player* | *Move* |
|---|---|
| A | 2 |
| B | 3 |
| A | 1, 6 |
| B | 8 |
| A | 4, 7 |
| B | 5 |
| B wins | |

***Example 2***

| *Player* | *Move* |
|---|---|
| A | 2, 5 |
| B | 7, 6 |
| A | 4 |
| B | 1 |
| A | 3, 8 |
| A wins | |

In this new game, there is a winning strategy for the second player. If the first player crosses out an even number, player B must do the same; if the first player crosses out an odd number, player B must do the same; and if the first player crosses out two numbers (one of the even and the other odd), player B must do the same. By "copying" the first player's moves, player B guarantees a win, because of making the last move.

## New Cross out-10

Although both 8 and 10 are even numbers, if player B tries to use the same strategy described for *New Cross out-8*, some problems will be experienced, as shown in the next example.

***Example***

| Player | Move |
|---|---|
| A | 2, 9 |
| B | 10, 3 |
| A | 1 |
| B | 5 |
| A | 4, 7 |
| B (*) | 6 |
| A | 8 |
| A wins | |

Player B tried to use the sequence of moves that the strategy includes, but was unable to copy player A's a third move (*) because there were no numbers left in the board. Why did it happen? The board in Figure 8 gives a hint.

In this board, the number of odd numbers, as well as the number of even numbers, is odd. That is the reason why player B could not copy player A's third move.

In this game, the first player has an advantage. If player A crosses out an odd number and an even one, player B is left in the situation of being the first player in *New-Cross out* with eight numbers (four even numbers, and four odd numbers). The first player has a winning strategy for this game, which is:

- During the first move, cross out two numbers (an odd number and an even one)
- In all the subsequent moves, "copy" all the moves made by player B.

It is important to note that the fact that the numbers are consecutive was not used in the analysis of the game. The same strategy will be relevant with any ten natural numbers such that the number of odd numbers, as well as the number of even numbers, is odd. This observation may lead students to identify more cases in which this strategy also leads to a win.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

Figure 8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

Figure 9

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

Figure 10

## New Cross out-9

In this case, Figure 9 shows a board in which the number of even numbers is even but the number of odd numbers is odd.

If the first player crosses out an odd number on the opening move, a win is guaranteed because the second player is left in the situation of being first player in *New Cross out* with eight numbers (four even numbers, and four odd numbers). If player A does not cross out an odd number, then player B can win, as is shown in the next example.

***Example***

| Player | Move |
|---|---|
| A | 2 |
| B | 8, 3 |
| A | 1, 6 |
| B | 4, 9 |
| A | 7 |
| B | 5 |
| B wins | |

To win in *New Cross out-9*, a player must:

- Play first and cross out an odd number.
- In all the subsequent moves, "copy" all the moves done by the other player.

Figure 10 shows a board with nine consecutive natural numbers, starting from 2.

In this case, to win a player must play first and cross out any even number because the number of even numbers is five. If the first player then follows the moves described by the strategy, a win will follow.

### Generalisation

To verify whether the winning strategy for the game *New Cross out* is understood, there are some more examples to check.

In order to win in the *New Cross-out* game, players must pay attention to two numbers: the number of odd numbers in the board (o) and the number of even numbers in the board (e).

- If both (o) and (e) are even, then there is a winning strategy for the second player.
- If both (o) and (e) are odd, then there is a winning strategy for the first player: start by crossing out an odd number and an even number.
- If only (o) is odd, then there is a winning strategy for the first player: start by crossing out any odd number.
- If only (e) is odd, then there is a winning strategy for the first player: start by crossing any even number.

| Numbers on the board | Board size | Will you play A? | Will you play B? | What numbers will you cross out in your first move? |
|---|---|---|---|---|
| 1, 2, 3, 4, 5, 6, 7 | 7 | Yes | | Any even number |
| 1, 3, 5, 7, 9, 11, 13 | 7 | Yes | | Any number |
| 2, 4, 6, 8, 10, 12, 14 | 7 | Yes | | Any number |
| 1, 2, 3, 5, 8, 10, 14 | 7 | Yes | | Any odd number |
| 2, 4, 6, 8, 10, 12 | 6 | | Yes | Any number |
| 5, 10, 15, 20, 25, 30 | 6 | Yes | | Any odd number + any even number |
| 3, 6, 9, 12, 15, 18, 21, 24 | 8 | | Yes | "Copy" the other player's moves |
| 3, 6, 9, 15, 18, 21, 24, 27 | 8 | Yes | | Any odd number + any even number |
| 1, 3, 5, 7, 9, 11, 13, 15 | 8 | | Yes | Any number |

## Concluding remarks

The mathematical games described in this article foster mathematical reasoning based on elementary and central mathematical contents like the ideas of parity and symmetry, and do not require sophisticated equipment. They may also constitute an appropriate opportunity to introduce students to the logic embedded in "If… then" statements. They can also be introduced to the idea that when they formulate their strategies in terms of, "If you do this, I will do that," they may notice that they need also to decide what to do, "If you don't do this". For school students, the process of discovering a winning strategy may constitute a genuine instance of the use of problem solving heuristics. In order to advance this learning episode into a complete mathematical activity, the discovery of a winning strategy and its following refinement may be followed by a justification step, by fostering the use of arguments appropriate to the level of the students who are playing the game. The mathematical communication and the social interactions among students promoted by this type of game may be another justification to include mathematical games in the class. These interactions can also expose them to the value of taking turns and sharing, and even to experience winning and losing.

From a teacher's perspective, the presentation sequence of the games may be central in order to facilitate the application of problem solving strategies like the use of analogy, reduction to known cases, and generalisation as well as the formulation of new games, as a result of the implementation of the "What if…?" problem posing technique.

I definitely recommend the use of this type of strategy game in schools as a tool for promoting mathematical communication and for enhancing fundamental mathematical activities like observation, interpretation of data, formulation of conjectures, generalisation, explanation, and even proving.

## Reference


Gardner, M. (1961). *Mathematical Puzzles*. New York: Thomas Y. Crowell Co.



**Greisy Winicki Landman**
California State Polytechnic University, USA
email: greisyw@csupomona.edu